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Optimal Control Problems via Exact Penalty Functions

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Abstract. The nonsmoothness is viewed by many people as at least an undesirable (if not unavoidable) property. Our aim here is to show that recent developments in Nonsmooth Analysis (especially in Exact Penalization Theory) allow one to treat successfully even some quite "smooth" problems by tools of Nonsmooth Analysis and Nondifferentiable Optimization. Our approach is illustrated by one Classical Control Problem of finding optimal parameters in a system described by ordinary differential equations.

Key words: Optimal control, Exact penalization, Directional differentiability, Subdifferentiability, Necessary optimality conditions, Nonsmooth analysis.

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1. Introduction

The problem of reducing a constrained mathematical programming problem to an unconstrained one has been given a great deal of attention. In most cases such a reduction is performed with the help of so-called penalty functions. At present the Theory of Penalization is well developed and widely used (see, e.g., [1–4]).

The exact penalization approach is most interesting and elegant but it generally requires solving a nonsmooth problem even if the original one was smooth. However, recent developments in Nondifferentiable Optimization give some hope that these difficulties will be overcome. To be able to reduce a constrained optimization problem to an unconstrained one via exact penalization it is suitable to represent the constraining set in the form of equality, where the function describing the set must satisfy some conditions on its directional derivatives (or, in general, on its generalized directional derivatives) (see [3, 5]).

In the present paper we show how to describe the constraints – given in the form of differential equations – by a (nonsmooth) functional whose directional derivatives satisfy the required properties (see Section 2). In Section 3 we treat one parametric optimization problem. This problem is reduced to a nonsmooth unconstrained optimization problem. It makes it possible to construct a numerical algorithm for the unconstrained optimization problem just allowing one to solve the original parametric optimization problem. Then, by making use of necessary optimality conditions (for a nonsmooth problem) it is shown that the conditions we obtain are equivalent to the well-known ones.

2. Systems of differential equations depending on parameters

Let $x \in \mathbb{R}^n$, $A \in \mathbb{R}^m$, $t \in [0,T], T > 0$ fixed, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ be differentiable with respect to x and A. The functions f, $\partial f/\partial x$, $\partial f/\partial A$ are assumed to be continuous on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. Consider the following system of differential equations, depending on the parameter $A \in \mathbb{R}^m$:

$$\dot{x} = f(x, A, t), \tag{1}$$

$$x(0) = x_0. \tag{2}$$

Let $C[0,T]$ be the class of n-dimensional vector functions $z(t)$ continuous on $[0,T]$. Consider the set

$$\Omega := \{[z,A] | z \in C[0,T], A \in \mathbb{R}^m : \varphi(z,A) = 0\}, \tag{3}$$

where

$$\varphi(z,A) := \left[\int_0^T \left(z(t) - f\left(x_0 + \int_0^t z(\tau)\,d\tau, A, t\right)\right)^2 dt\right]^{1/2}. \tag{4}$$

Note that $\varphi(z,A) \geq 0\ \forall z \in C[0,T]$, $\forall A \in \mathbb{R}^m$. If $[z,A] \in \Omega$, then obviously the function $x(t) = x_0 + \int_0^t z(\tau)\,d\tau$ satisfies (1), (2), and vice versa, if $x(t)$ is a solution of (1), (2) then $[z,A] \in \Omega$ (with $z(t) = f(x,A,t)$). Thus, the problem of finding a solution of (1), (2) for some fixed $A \in \mathbb{R}^m$ is equivalent to finding a $z \in C[0,T]$ such that $\varphi(z,A) = 0$.

Now let us study the differentiability properties of the function φ. First of all, consider the case:

$$\varphi(z,A) > 0. \tag{5}$$

Let $g := [v,q]$, where $v \in C[0,T], q \in \mathbb{R}^m$. Put $\|g\| := \max\{\|v\|, \|q\|\}$ where $\|v\| := [\int_0^T (v(t))^2\,dt]^{1/2}$, $\|q\| := \sqrt{q^2}$. Here, as usual, $a^2 := (a,a)$. The pair g will be called a direction (in the space $C[0,T] \times \mathbb{R}^m$). Let us find the directional derivative of φ at some point $[z,A]$ satisfying (5) in a direction g. By definition

$$\varphi'(z,A;g) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}[\varphi(z + \alpha v, A + \alpha q) - \varphi(z,A)]. \tag{6}$$

We shall prove, that this limit does exist, and find its value.

LEMMA 1. *If $\varphi(z,A) > 0$ (i.e., $[z,A] \notin \Omega$), then the function φ is Gâteaux differentiable at $[z,A]$.*

Proof. We have

$$\begin{aligned}
&\varphi(z+\alpha v, A+\alpha q)\\
&\quad=\left[\int_0^T\left(z(t)+\alpha v(t)-f\left(x_0+\int_0^t(z(\tau)+\alpha v(\tau))d\tau,\ A+\alpha q,t\right)\right)^2 dt\right]^{1/2}\\
&\quad=\left\{\int_0^T\left[z(t)-f\left(x_0+\int_0^t z(\tau)\,d\tau,A,t\right)\right.\right.\\
&\qquad\left.\left.+\ \alpha\left[v(t)-\frac{\partial f(t)}{\partial x}\int_0^t v(\tau)\,d\tau-\frac{\partial f(t)}{\partial A}\,q\right]+o(\alpha)\right]^2 dt\right\}^{1/2}. \qquad (7)
\end{aligned}$$

Here $\partial f(t)/\partial x = \partial f(x(t),A,t)/\partial x$, $\partial f(t)/\partial A = \partial f(x(t),A,t)/\partial A$. Since $h(\alpha) = [(a+\alpha b)^2]^{1/2} = (a^2+2\alpha(a,b)+\alpha^2 b^2)^{1/2}$, then $h'(0) = (a,b)/\|a\|$. Therefore (7) implies

$$\varphi'(z,A;g) = \int_0^T\left(w(t), v(t)-\frac{\partial f(t)}{\partial x}\int_0^t v(\tau)\,d\tau-\frac{\partial f(t)}{\partial A}\,q\right)dt, \qquad (8)$$

where

$$w(t) := \frac{1}{\varphi(z,A)}\left(z(t)-f\left(x_0+\int_0^t z(\tau)\,d\tau,A,t\right)\right). \qquad (9)$$

It is clear that

$$\|w\| := \left[\int_0^T (w(t))^2\,dt\right]^{1/2} = 1. \qquad (10)$$

In (8) let us set (an * as apex denotes transposition):

$$\begin{aligned}
B &:= \int_0^T\left(\frac{\partial f(t)}{\partial x}\int_0^t v(\tau)\,d\tau, w(t)\right)dt\\
&= \int_0^T\left(\int_0^t v(\tau)\,d\tau, \left(\frac{\partial f(t)}{\partial x}\right)^* w(t)\right)dt. \qquad (11)
\end{aligned}$$

Let us integrate (11) by parts:

$$u = \int_0^t v(\tau)\,d\tau, \quad du = v(t)\,dt,$$

$$\begin{aligned}
d\bar v &= \left(\frac{\partial f(t)}{\partial x}\right)^* w(t)\,dt, \quad \bar v = \int_0^t\left(\frac{\partial f(\tau)}{\partial x}\right)^* w(\tau)\,d\tau,\\
B &= \left(\int_0^T v(t)\,dt, \int_0^T\left(\frac{\partial f(t)}{\partial x}\right)^* w(t)\,dt\right) -
\end{aligned}$$

$$- \int_0^T \left(v(t), \int_0^t \left(\frac{\partial f(\tau)}{\partial x} \right)^* w(\tau)\, d\tau \right) dt$$

$$= \int_0^T \left(v(t), \int_t^T \left(\frac{\partial f(\tau)}{\partial x} \right)^* w(\tau)\, d\tau \right) dt\,. \tag{12}$$

Substituting (12) in (8) we get

$$\varphi'(z, A; g) \;=\; \int_0^T \left(v(t), w(t) - \int_t^T \left(\frac{\partial f(\tau)}{\partial x} \right)^* w(\tau)\, d\tau \right) dt$$

$$- \left(\int_0^T \left(\frac{\partial f(t)}{\partial A} \right)^* w(t)\, dt, q \right) = (\nabla \varphi, g), \tag{13}$$

where

$$\nabla \varphi = \left[w(t) - \int_t^T \left(\frac{\partial f(\tau)}{\partial x} \right)^* w(\tau)\, d\tau, - \int_0^T \left(\frac{\partial f(t)}{\partial A} \right)^* w(t)\, dt \right] \tag{14}$$

and $(\nabla \varphi, g)$ will be referred to as "the scalar product" of $\nabla \varphi$ and g. Since (13) is linear in g and the complement of Ω is open, we conclude that φ is Gâteaux differentiable at $[z, A]$ with "the gradient" $\nabla \varphi$ (in the space $C[0, T] \times \mathbb{R}^m$). This completes the proof. □

LEMMA 2. *There exists* $a > 0$ *such that*

$$\min_{\|g\|=1} (\nabla \varphi, g) \leq -a < 0 \quad \forall [z, A] \notin \Omega. \tag{15}$$

Proof. Let us prove, first of all, that

$$\nabla \varphi \neq \mathbf{0}. \tag{16}$$

Here **0** is the zero element of the space $C[0, T] \times \mathbb{R}^m$. Assuming the opposite, we have:

$$w(t) - \int_t^T \left(\frac{\partial f(\tau)}{\partial x} \right)^* w(\tau)\, d\tau = 0_n \quad \forall t \in [0, T]. \tag{17}$$

Then (17) implies $w(t) = 0_n\ \forall t \in [0, T]$ which contradicts (10). Thus (16) holds. Suppose now that (15) is invalid. Then, there exists a sequence $[z_k, A_k]$ such that

$$[z_k, A_k] \notin \Omega, \quad \nabla \varphi_k \to \mathbf{0}, \tag{18}$$

where

$$\nabla \varphi_k = \left[w_k(t) - \int_t^T \left(\frac{\partial f_k(\tau)}{\partial x} \right)^* w_k(\tau)\, d\tau, - \int_0^T \left(\frac{\partial f_k(t)}{\partial A} \right)^* w_k(t)\, dt \right],$$

$$\frac{\partial f_k(t)}{\partial x} = \frac{\partial f(x_k(t), A_k, t)}{\partial x}, \; \frac{\partial f_k(t)}{\partial A} = \frac{\partial f(x_k(t), A_k, t)}{\partial A},$$

$$x_k(t) = x_0 + \int_0^t z_k(\tau) d\tau,$$

$$w_k(t) = \frac{1}{\varphi(z_k, A_k)} \left(z_k(t) - f\left(x_0 + \int_0^t z_k(\tau)\, d\tau, A_k, t \right) \right).$$

Note that

$$\|w_k\| = 1. \tag{19}$$

(18) implies

$$\|h_k\| \to 0, \tag{20}$$

where

$$h_k(t) = w_k(t) - \int_t^T \left(\frac{\partial f_k(\tau)}{\partial x} \right)^* w_k(\tau)\, d\tau. \tag{21}$$

Relations (20) and (21) yield (due to the continuous dependence of the solutions of integral equations on the right-hand sides) $\|w_k\| \to 0$ which contradicts (19). This completes the proof. □

Now consider the case where $\varphi(z, A) = 0$. Note that

$$\varphi(z, A) = \max_{\|\bar{v}\|=1} \int_0^T \left(z(t) - f\left(x_0 + \int_0^t z(\tau)\, d\tau, A, t \right), \bar{v}(t) \right) dt. \tag{22}$$

If $\varphi(z, A) = 0$, then $h(t) = z(t) - f(x_0 + \int_0^t z(\tau)\, d\tau, A, t) = 0\, \forall t \in [0, T]$. Since

$$z(t) + \alpha v(t) - f\left(x_0 + \int_0^t (z(\tau) + \alpha v(\tau))\, d\tau, A + \alpha q, t \right) =$$
$$= h(t) + \alpha \left[v(t) - \frac{\partial f(t)}{\partial x} \int_0^t v(\tau)\, d\tau - \frac{\partial f(t)}{\partial A} q \right] + o(\alpha),$$

then (see (22)):

$$\varphi'(z, A; g) = \max_{\|\bar{v}\|=1} \int_0^T \left(\bar{v}(t), v(t) - \frac{\partial f(t)}{\partial x} \int_0^t v(\tau)\, d\tau - \frac{\partial f(t)}{\partial A} q \right) dt. \tag{23}$$

Using the same procedure as in (11)–(12), from (23) we get:

$$\varphi'(z, A, g) = \max_{\|\bar{v}\|=1} \left[\int_0^T \left(v(t), \bar{v}(t) - \int_t^T \left(\frac{\partial f(t)}{\partial x} \right)^* \bar{v}(\tau)\, d\tau \right) dt \right.$$
$$\left. - \left(\int_0^T \left(\frac{\partial f(t)}{\partial A} q \right)^* \bar{v}(t)\, dt, q \right) \right]. \tag{24}$$

(23) and (24) show that the following proposition holds.

LEMMA 3. *If* $\varphi(A, z) = 0$, *then the function* φ *is directionally differentiable at* $[A, z]$; *it is even subdifferentiable, i.e.*

$$\varphi'(z, A; g) = \max_{G \in \partial_\varphi(z,A)} (G, g), \tag{25}$$

where

$$\begin{aligned} \partial\varphi(z, A) &= \{G = [v^*, q^*] : v^* \in C[0, T], q^* \in \mathbb{R}^m, v^*(t) = \bar{v}(t) \\ &\quad - \int_t^T \left(\frac{\partial f(t)}{\partial A}\right)^* \bar{v}(\tau)\, d\tau, \\ q^* &= -\int_0^T \left(\frac{\partial f(t)}{\partial A}\right)^* \bar{v}(t)\, dt, \bar{v} \in C[0, T], \|\bar{v}\| \leq 1 \Bigg\}. \end{aligned} \tag{26}$$

3. Parametric optimization problems: the case of a smooth functional

To illustrate our approach let us consider the problem of minimizing the functional

$$\mathcal{I}(A) = \int_0^T F(x(t, A)) dt, \tag{27}$$

where $x(t, A)$ is a solution of (1), (2) with $A \in \mathbb{R}^m$, and $F(x)$ is a smooth function. It follows from Section 2 that the above problem is equivalent to the problem of minimizing the functional

$$\phi(z, A) = \int_0^T F\left(x_0 + \int_0^t z(\tau)\, d\tau\right)\, dt \tag{28}$$

subject to the constraint

$$\varphi(z, A) = 0. \tag{29}$$

The functional $\phi(z, A)$ does not depend on A explicitly (it depends on it implicitly via (29)). It is easy to see that

$$\begin{aligned} \phi'(z, A; g) &= \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \left[\int_0^T F\left(x_0 + \int_0^t (z(\tau) + \alpha v(\tau))\, d\tau\right)\, dt - \phi(z, A)\right] \\ &= \int_0^T \left(\frac{\partial F(x(t))}{\partial x}, v(t)\right)\, dt \end{aligned}$$

i.e. ϕ is Gâteaux differentiable and its "gradient" (in the space $C[0, T] \times \mathbb{R}^m$) is

$$\nabla\phi(z, A) = \left[\frac{\partial F(x(t))}{\partial x}, 0_m\right]. \tag{30}$$

Arguing as in [3–5] we are able to prove the following:

THEOREM 1. *If φ is Lipschitz on $C[0,T] \times \mathbb{R}^m$, then there exists a $\lambda_0 \geq 0$ such that for any $\lambda \geq \lambda_0$ the set of minimizers of ϕ on the set $\Omega = \{[z,A] | \varphi(z,A) = 0\}$ coincides with the set of minimizers of the function*

$$\varphi_\lambda(z,A) = \phi(z,A) + \lambda\varphi(z,A) \tag{31}$$

on the entire space $C[0,T] \times \mathbb{R}^m$.

Thus, if $[z^*, A^*]$ is a minimizer of $\psi_\lambda(z,A)$ (for $\lambda \geq \lambda_0$), then $\varphi(z^*, A^*) = 0$ and ϕ attains its minimum on Ω at $[z^*, A^*]$. This also implies that the function $x^*(t) = x_0 + \int_0^t z^*(\tau)\,d\tau$ satisfies the system of differential equations

$$\dot{x}(t) = f(x(t), A^*, t), \quad x(0) = x_0,$$

and the functional $\mathcal{I}(A)$ attains its minimum at A^*.

The function $\psi_\lambda(z,A)$ is subdifferentiable and its subdifferential (see [6]) is

$$\partial\psi_\lambda(z^*, A^*) = \nabla\phi(z^*, A^*) + \lambda\partial\varphi(z^*, A^*), \tag{32}$$

where $\nabla\phi$ is defined by (30) and $\partial\varphi$ by (26) (since $\varphi(z^*, A^*) = 0$).

Applying the necessary optimality condition (see [6]) we get

$$\mathbf{0} \in \partial\psi_\lambda(z^*, A^*). \tag{33}$$

Thus, it follows from (33) that there exists a $\bar{v} \in C[0,T]$ such that $\|\bar{v}\| \leq 1$,

$$\frac{\partial F(x^*(t))}{\partial x} + \lambda\left[\bar{v}(t) - \int_t^T \left(\frac{\partial f(\tau)}{\partial x}\right)^* \bar{v}(\tau)\,d\tau\right] = 0_n \quad \forall t \in [0,T], \tag{34}$$

$$-\lambda\int_0^T \left(\frac{\partial f(t)}{\partial A}\right)^* \bar{v}(t)\,dt = 0_m\,. \tag{35}$$

Here

$$\frac{\partial f(t)}{\partial x} = \frac{\partial f(x^*(t), A^*, t)}{\partial x}, \quad \frac{\partial f(t)}{\partial A} = \frac{\partial f(x^*(t), A^*, t)}{\partial A}.$$

Replacing $\lambda\bar{v}(t)$ by $v(t)$ we conclude from (34) and (35) that, if $x^*(t) = x(t, A^*)$ is a minimizer of (27), then there exists a vector function $v(t) \in C[0,T]$ such that

$$\frac{\partial F(x^*(t))}{\partial x} + v(t) - \int_t^T \left(\frac{\partial f(\tau)}{\partial x}\right)^* v(\tau)\,d\tau = 0 \quad \forall t \in [0,T] \tag{36}$$

and

$$\int_0^T \left(\frac{\partial f(t)}{\partial A}\right)^* v(t)\,dt = 0_m. \tag{37}$$

If $F(x)$ is twice continuously differentiable, then (36) can be rewritten in the following "differential" form

$$\dot{v}(t) = -\left(\frac{\partial f(t)}{\partial x}\right)^* v(t) - \frac{d}{dt}\left(\frac{\partial F(x^*(t))}{\partial x}\right), \tag{38}$$

$$v(T) = -\frac{\partial F(x^*(\tau))}{\partial x}. \tag{39}$$

The function v is uniquely defined by (36) (or, equivalently, by (38)–(39)).

Finally, we can state the following necessary optimality condition.

THEOREM 2. *If $A^* \in \mathbb{R}^m$ is a minimizer of $\mathcal{I}(A)$ subject to the system of differential equations (1)–(2), then the function $v(t) \in C[0,T]$ defined by (38) and (39) satisfies (37).*

Remark 1. It is necessary to note that the idea of reducing the problem of minimizing the functional (27) on the solutions of the system (1), (2) to an infinite sequence of unconstrained smooth problems (using in (3) the function $\varphi^2(z,A)$ instead of $\varphi(z,A)$) was proposed by A.V. Balakrishnan [7] and successfully used by G. Di Pillo and L. Grippo [8] (see also [9]).

Remark 2. Of course, Theorem 2 is well-known from Control Theory (see [10]). The most interesting here is that the problem of finding "optimal" parameters A (the problem of minimizing (27) is a constrained optimization problem since A is supposed to satisfy (1), (2)) is reduced to an unconstrained optimization problem (see Theorem 1) and now one can use numerical methods for unconstrained (but Nonsmooth) optimization (see [6]).

Remark 3. There is no difficulty to conceive the application of the above approach to the case where the functional (27) is itself nonsmooth. Then, one obtains some new results which do not follow from the Classical Optimal Control Theory. The way is open to do this.

Remark 4. The problem of minimizing the functional (27) on the solutions of the system (1), (2) (problem P) can also be formulated in the following equivalent form: minimize (with respect to $[z,A]$) the functional (28) subject to the constraints

$$z(t) - f\left(x_0 + \int_0^t z(\tau)\,d\tau, A, t\right) = 0 \quad \forall t \in [0,T]. \tag{40}$$

Here $x(t) = x_0 + \int_0^t z(\tau)\,d\tau$. Now the following classic problem can be considered: given a solution $[x^*(t), A^*]$ of problem P, does a function $\lambda(t) \in C[0,T]$ exist such that the point $[z^*(t), A^*]$ is a critical point of the Lagrangian function

$$L(\lambda, z, A) := \int_0^T \left[F\left(x_0 + \int_0^t z(\tau) d\tau\right)\right.$$

$$+\left(\lambda(t), z(t) - f\left(x_0 + \int_0^t z(\tau)\,d\tau), A, t\right)\right)\Big]\ dt?$$

The answer is positive and follows from Theorem 2: such a (Lagrangian multiplier) function is the function $\lambda(t) = v(t)$, satisfying (38)–(39).

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